

Solving the At-Most-Once Problem with Nearly Optimal Effectiveness

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Abstract

We present and analyze a wait-free deterministic algorithm for solving the at-most-once problem: how m shared-memory fail-prone processes perform asynchronously n tasks at most once. Our algorithmic strategy provides for the first time nearly optimal effectiveness, which is a measure that expresses the total number of tasks completed in the worst case. The effectiveness of our algorithm equals $n - 2m + 2$. This is up to an additive factor of m close to the known effectiveness upper bound $n - m + 1$ over all possible algorithms and improves on the previously best known deterministic solutions that have effectiveness only $n - \log m \cdot o(n)$. We also present an iterated version of our algorithm that for any $m = O(\sqrt[3+\epsilon]{n/\log n})$ is both effectiveness-optimal and work-optimal, for any constant $\epsilon > 0$. We then employ this algorithm to provide a new explicit algorithmic solution for the Write-All problem which is work optimal for any $m = O(\sqrt[3+\epsilon]{n/\log n})$ improving the previously best known result of $m = O(\sqrt[4]{n/\log n})$.

Keywords: At-most-once problem, Write-All, I/O automata, asynchronous shared memory.

1 Introduction

The *at-most-once problem* for asynchronous shared memory systems was introduced by Kentros et al. [15] as the problem of performing a set of n jobs by m fail-prone processes while maintaining at-most-once semantics.

The *at-most-once* semantic for object invocation ensures that an operation accessing and altering the state of an object is performed no more than once. This semantic is among the standard semantics for remote procedure calls (RPC) and method invocations and it provides important means for reasoning about the safety of critical applications. Uniprocessor systems may trivially provide solutions for at-most-once semantics by implementing a central schedule for operations. The problem becomes very challenging for autonomous processes in a system with concurrent invocations on multiple objects.

Perhaps the most important question in this area is devising algorithms for the at-most-once problem with good *effectiveness*. The complexity measure of effectiveness [15] describes the number of jobs completed (at-most-once) by an implementation, as a function of the overall number of jobs n , the number of processes m , and the number of crashes f . The only deterministic solutions known exhibit very low effectiveness $\left(n^{\frac{1}{\log m}} - 1\right)^{\log m}$ (see [15]) which for most choices of the parameters is very far from optimal (unless $m = O(1)$). Contrary to this, the present work presents the first deterministic algorithm for the at-most-once problem which is optimal up to additive factors of m . Specifically our effectiveness is

$n - (2m - 2)$ which comes close to an additive factor of m to the known ([15]) upper bound over all possible algorithms for effectiveness $n - m + 1$. We also demonstrate how to construct an algorithm which has effectiveness $n - O(m^2 \log n \log m)$ and work complexity $O(n + m^{3+\epsilon} \log n)$, and is both effectiveness and work optimal when $m = O(\sqrt[3+\epsilon]{n/\log n})$, for any constant $\epsilon > 0$. Work complexity counts the total number of basic operations performed by the processes. Finally we show how to use this algorithm in order to solve the *Write-All* problem [14] with work complexity $O(n + m^{3+\epsilon} \log n)$, which improves on the best known explicit result by Malewicz [24] that has work complexity $O(n + m^4 \log n)$.

Related Work: A wide range of works study at-most-once semantics in a variety of settings. At-most-once message delivery [5, 18, 21, 27] and at-most-once semantics for RPC [4, 19, 20, 21, 25], are two areas that have attracted a lot of attention. Both in at-most-once message delivery and RPCs, we have two entities (sender/client and receiver/server) that communicate by message passing. Any entity may fail and recover and messages may be delayed or lost. In the first case one wants to guarantee that duplicate messages will not be accepted by the receiver, while in the case of RPCs, one wants to guarantee that the procedure called in the remote server will be invoked at-most-once [26].

In Kentros et al. [15], the at-most-once problem for asynchronous shared memory systems and the correctness properties to be satisfied by any solution were defined. The first algorithms that solve the at-most-once problem were provided and analyzed. Specifically they presented two algorithms that solve the at-most-once problem for two processes with optimal effectiveness and a multi-process algorithm, that employs a two-process algorithm as a building block, and solves the at-most-once problem with effectiveness $n - \log m \cdot o(n)$ and work complexity $O(n + m \log m)$. Subsequently Hillel [13] provided a probabilistic algorithm in the same setting with optimal effectiveness and expected work complexity $O(nm^2 \log m)$ by employing a probabilistic multi-valued consensus protocol as a building block.

Di Crescenzo and Kiayias in [6] (and later Fitzi et al. [8]) demonstrate the use of the semantic in message passing systems for the purpose of secure communication. Driven by the fundamental security requirements of *one-time pad* encryption, the authors partition a common random pad among multiple communicating parties. Perfect security can be achieved only if every piece of the pad is used at most once. The authors show how the parties maintain security while maximizing efficiency by applying at-most-once semantics on pad expenditure.

One can also relate the at-most-once problem to the consensus problem [7, 12, 23, 17]. Indeed, consensus can be viewed as an at-most-once distributed decision. Another related problem is process renaming, see Attiya *et al.* [2] where each process identifier should be assigned to at most one process.

The at-most-once problem has also many similarities with the *Write-All* problem for the shared memory model [1, 10, 14, 16, 24]. First presented by Kanellakis and Shvartsman [14], the *Write-All* problem is concerned with performing each task *at-least-once*. Most of the solutions for the *Write-All* problem, exhibit super-linear work when $m \ll n$. Malewicz [24] was the first to present a solution for the *Write-All* problem that has linear work for a non-trivial number of processors. The algorithm presented by Malewicz [24] has work $O(n + m^4 \log n)$ and uses test-and-set operations. Later Kowalski and Shvartsman [16] presented a solution for the *Write-All* problem that for any constant ϵ has work $O(n + m^{2+\epsilon})$. Their algorithm uses a collection of q permutations with contention $O(q \log q)$ for a properly choose constant q .

We note that the at-most-once problem becomes much simpler when shared-memory is supplemented by some type of read-modify-write operations. For example, one can associate a *test-and-set* bit with each task, ensuring that the task is assigned to the only process that successfully sets the shared bit. An effectiveness optimal implementation can then be easily obtained from any *Write-All* solution. Thus, in this paper we deal only with the more challenging setting where algorithms use atomic read/write registers.

Contributions: In this paper we present and analyze the algorithm KK_β that solves the at-most-once problem. The algorithm is parametrized by $\beta \geq m$ and has effectiveness $n - \beta - m + 2$. If $\beta < m$ the correctness of the algorithm is still guaranteed, but the termination of the algorithm cannot be guaranteed.

For $\beta = m$ the algorithm has optimal effectiveness of $n - 2m + 2$ up to an additive factor of m . Note that the upper bound for the effectiveness of any algorithm is $n - f$ [15], where $f \leq m - 1$ is the number of failures in the system. We further prove that for $\beta \geq 3m^2$ the algorithm has work complexity $O(nm \log n \log m)$. We use algorithm KK_β with $\beta = 3m^2$, in order to construct an iterated version of our algorithm which for any constant $\epsilon > 0$, has effectiveness of $n - O(m^2 \log n \log m)$ and work complexity $O(n + m^{3+\epsilon} \log n)$. This is both effectiveness-optimal and work-optimal for any $m = O(\sqrt[3+\epsilon]{n/\log n})$. We note that our solutions are deterministic and assume worst-case behavior. In the probabilistic setting Hillel [13] shows that optimal effectiveness can be achieved with expected work complexity $O(nm^2 \log m)$.

We then demonstrate how to use the iterated version of our algorithm in order to solve the Write-All problem with work complexity $O(n + m^{3+\epsilon} \log n)$ for any constant $\epsilon > 0$. Our solution improves on the algorithm of Malewicz [24], which is the best known explicit result, in two ways. Firstly our solution is work optimal for a wider range of m , namely for any $m = O(\sqrt[3+\epsilon]{n/\log n})$ compared to the $m = O(\sqrt[4]{n/\log n})$ of Malewicz. Secondly our solution does not assume the test-and-set primitive used by Malewicz [24], and relies *only* on atomic read/write memory. Note that there is a Write-All algorithm due to Kowalski and Shvartsman [16], which is work optimal for a wider range of processors m than our algorithm, specifically for $m = O(\sqrt[2+\epsilon]{n})$. However, their algorithm uses a collection of q permutations with contention $O(q \log q)$, while it is not known to date how to construct such permutations in polynomial time. Thus their result is so far existential, while ours is explicit.

2 Model, Definitions, and Efficiency

We define our model, the at-most-once problem, and measures of efficiency.

2.1 Model and Adversary

We model a multi-processor as m asynchronous, crash-prone processes with unique identifiers from some set \mathcal{P} . Shared memory is modeled as a collection of atomic read/write memory cells, where the number of bits in each cell is explicitly defined. We use the *Input/Output Automata* formalism [22, 23] to specify and reason about algorithms; specifically, we use the *asynchronous shared memory automaton* formalization [9, 23]. Each process p is defined in terms of its states states_p and its actions acts_p , where each action is of the type *input*, *output*, or *internal*. A subset $\text{start}_p \subseteq \text{states}_p$ contains all the start states of p . Each shared variable x takes values from a set V_x , among which there is init_x , the initial value of x .

We model an algorithm A as a composition of the automata for each process p . Automaton A consists of a set of states $\text{states}(A)$, where each state s contains a state $s_p \in \text{states}_p$ for each p , and a value $v \in V_x$ for each shared variable x . Start states $\text{start}(A)$ is a subset of $\text{states}(A)$, where each state contains a start_p for each p and an init_x for each x . The actions of A , $\text{acts}(A)$ consists of actions $\pi \in \text{acts}_p$ for each process p . A transition is the modification of the state as a result of an action and is represented by a triple (s, π, s') , where $s, s' \in \text{states}(A)$ and $\pi \in \text{acts}(A)$. The set of all transitions is denoted by $\text{trans}(A)$. Each action in $\text{acts}(A)$ is performed by a process, thus for any transition (s, π, s') , s and s' may differ only with respect to the state s_p of process p that invoked π and potentially the value of the shared variable that p interacts with during π . We also use triples $(\{\text{vars}_s\}, \pi, \{\text{vars}_{s'}\})$, where vars_s and $\text{vars}_{s'}$ are subsets of variables in s and s' respectively, as a shorthand to describe transitions without having to specify s and s' completely; here vars_s and $\text{vars}_{s'}$ contain only the variables whose value changes as the result of π , plus possibly some other variables of interest.

An *execution* fragment of A is either a finite sequence, $s_0, \pi_1, s_1, \dots, \pi_r, s_r$, or an infinite sequence, $s_0, \pi_1, s_1, \dots, \pi_r, s_r, \dots$, of alternating states and actions, where $(s_k, \pi_{k+1}, s_{k+1}) \in \text{trans}(A)$ for any $k \geq 0$. If $s_0 \in \text{start}(A)$, then the sequence is called an *execution*. The set of executions of A is $\text{execs}(A)$. We say that execution α is *fair*, if α is finite and its last state is a state of A where no locally controlled action is enabled, or α is infinite and every locally controlled action $\pi \in \text{acts}(A)$ is performed infinitely many times or there are infinitely many states in α where π is disabled. The set of fair executions of A is $\text{fairexecs}(A)$.

An execution fragment α' *extends* a finite execution fragment α of A , if α' begins with the last state of α . We let $\alpha \cdot \alpha'$ stand for the execution fragment resulting from concatenating α and α' and removing the (duplicated) first state of α' .

For two states s and s' of an execution fragment α , we say that state s *precedes* state s' and we write $s < s'$ if s appears before s' in α . Moreover we write $s \leq s'$ if state s either precedes state s' in α or the states s and s' are the same state of α . We use the term precedes and the symbols $<$ and \leq in a same way for the actions of an execution fragment. We use the term precedes and the symbol $<$ if an action π appears before a state s in an execution fragment α or if a state s appears before an action π in α . Finally for a set of states S of an execution fragment α , we define as $s_{max} = \max S$ the state $s_{max} \in S$, s.t. $\forall s \in S, s \leq s_{max}$ in α .

We model process crashes by action stop_p in $\text{acts}(A)$ for each process p . If stop_p appears in an execution α then no actions $\pi \in \text{acts}_p$ appear in α thereafter. We then say that process p *crashed*. Actions stop_p arrive from some unspecified external environment, called *adversary*. In this work we consider an *omniscient, on-line adversary* [14] that has complete knowledge of the algorithm executed by the processes. The adversary controls asynchrony and crashes. We allow up to $f < m$ crashes. We denote by $\text{fairexecs}_f(A)$ all fair executions of A with at most f crashes. Note that since the processes can only communicate through atomic read/write operations in the shared memory, all the asynchronous executions are linearizable. This means that concurrent actions can be mapped to an equivalent sequence of state transitions, where only one process performs an action in each transition, and thus the model presented above is appropriate for the analysis of a multi-process asynchronous atomic read/write shared memory system.

2.2 At-Most-Once Problem, Effectiveness and Complexity

We consider algorithms that perform a set of tasks, called *jobs*. Let A be an algorithm specified for m processes with ids from set $\mathcal{P} = [1 \dots m]$, and for n jobs with unique ids from set $\mathcal{J} = [1 \dots n]$. We assume that there are at least as many jobs as there are processes, i.e., $n \geq m$. We model the performance of job j by process p by means of action $\text{do}_{p,j}$. For a sequence c , we let $\text{len}(c)$ denote its length, and we let $c|_\pi$ denote the sequence of elements π occurring in c . Then for an execution α , $\text{len}(\alpha|_{\text{do}_{p,j}})$ is the number of times process p performs job j . Finally we denote by $F_\alpha = \{p | \text{stop}_p \text{ occurs in } \alpha\}$ the set of crashed processes in execution α . Now we define the number of jobs performed in an execution. Note here that we are borrowing most definitions from Kentros et al. [15].

Definition 2.1 For execution α we denote by $\mathcal{J}_\alpha = \{j \in \mathcal{J} | \text{do}_{p,j} \text{ occurs in } \alpha \text{ for some } p \in \mathcal{P}\}$. The total number of jobs performed in α is defined to be $\text{Do}(\alpha) = |\mathcal{J}_\alpha|$.

We next define the *at-most-once* problem.

Definition 2.2 Algorithm A solves the *at-most-once* problem if for each execution α of A we have $\forall j \in \mathcal{J} : \sum_{p \in \mathcal{P}} \text{len}(\alpha|_{\text{do}_{p,j}}) \leq 1$.

Definition 2.3 Let S be a set of elements with unique identifiers. We define as the *rank* of element $x \in S$ and we write $[x]_S$, the rank of x if we sort in ascending order the elements of S according to their identifiers.

Measures of Efficiency. We analyze our algorithms in terms of two complexity measures: *effectiveness* and *work*. Effectiveness counts the number of jobs performed by an algorithm in the worst case.

Definition 2.4 The *effectiveness* of algorithm A is: $E_A(n, m, f) = \min_{\alpha \in \text{fairexecs}_f(A)} (\text{Do}(\alpha))$, where m is the number of processes, n is the number of jobs, and f is the number of crashes.

A trivial algorithm can solve the at-most-once problem by splitting the n jobs in groups of size $\frac{n}{m}$ and assigning one group to each process. Such a solution has effectiveness $E(n, m, f) = (m - f) \cdot \frac{n}{m}$ (consider an execution where f processes fail at the beginning of the execution).

Work complexity measures the total number of basic operations (comparisons, additions, multiplications, shared memory reads and writes) performed by an algorithm. We assume that each internal or shared memory cell has size $O(\log n)$ bits and performing operations involving a constant number of memory cell costs $O(1)$. This is consistent with the way work complexity is measured in previous related work [14, 16, 24].

Definition 2.5 *The **work** of algorithm A , denoted by W_A , is the worst case total number of basic operations performed by all the processes of algorithm A .*

Finally we repeat here as a Theorem, Corollary 1 from Kentros et al. [15], that gives an upper bound on the effectiveness for any algorithm solving the at-most-once problem.

Theorem 2.1 *from Kentros et al. [15]*

For all algorithms A that solve the at-most-once problem with m processes and $n \geq m$ jobs in the presence of $f < m$ crashes it holds that $E_A(n, m, f) \leq n - f$.

3 Algorithm KK_β

Here we present algorithm KK_β , that solves the at-most-once problem. Parameter $\beta \in \mathbb{N}$ is the termination parameter of the algorithm. Algorithm KK_β is defined for all $\beta \geq m$. If $\beta = m$, algorithm KK_β has optimal up to an additive factor of m effectiveness. Note that although $\beta \geq m$ is not necessary in order to prove the correctness of the algorithm, if $\beta < m$ we cannot guarantee termination of algorithm KK_β .

The idea behind the algorithm KK_β (see Fig. 1) is quite intuitive and is based on an algorithm for renaming processes presented by Attiya *et al.* [2]. Each process p , picks a job i to perform, announces (by writing in shared memory) that it is about to perform the job and then checks if it is safe to perform it (by reading the announcements other processes made in the shared memory, and the jobs other processes announced they have performed). If it is safe to perform the job i , process p will proceed with the $do_{p,i}$ action and then mark the job completed. If it is not safe to perform i , p will release the job. In either case, p picks a new job to perform. In order to pick a new job, p reads from the shared memory and gathers information on which jobs are safe to perform, by reading the announcements that other processes made in the shared memory about the jobs they are about to perform, and the jobs other processes announced they have already performed. Assuming that those jobs are ordered, p splits the set of “free” jobs in m intervals and picks the first job of the interval with rank equal to p ’s rank. Note that since the information needed in order to decide whether it is safe to perform a specific job and in order to pick the next job to perform is the same, these steps are combined in the algorithm. In Figure 1, we use function $rank(\text{SET}_1, \text{SET}_2, i)$, that returns the element of set $\text{SET}_1 \setminus \text{SET}_2$ that has rank i . If SET_1 and SET_2 have $O(n)$ elements and are stored in some tree structure like *red-black tree* or some variant of *B-tree*, the operation $rank(\text{SET}_1, \text{SET}_2, i)$, costs $O(|\text{SET}_2| \log n)$ assuming that $\text{SET}_2 \subseteq \text{SET}_1$. We will prove that the algorithm has effectiveness $n - (\beta + m - 2)$. For $\beta = O(m)$ this effectiveness is asymptotically optimal for any $m = o(n)$. Note that by Theorem 2.1 the upper bound on effectiveness of the at-most-once problem is $n - f$, where f is the number of failed processes in the system. Next we present algorithm KK_β in more detail.

Shared Variables. *next* is an array with m elements. In the cell $next_q$ of the array process q announces the job it is about to perform. From the structure of algorithm KK_β , only process q writes in cell $next_q$. On the other hand any process may read cell $next_q$.

Shared Variables:

$next = \{next_1, \dots, next_m\}$, $next_q \in \{0, \dots, n\}$ initially 0
 $done = \{done_{1,1}, \dots, done_{m,n}\}$, $done_{q,i} \in \{0, \dots, n\}$ initially 0

Signature:

Input:
 $stop_p$, $p \in \mathcal{P}$
Output:
 $do_{p,j}$, $p \in \mathcal{P}$, $j \in \mathcal{J}$

Internal:
 $compNext_p$, $p \in \mathcal{P}$
 $check_p$, $p \in \mathcal{P}$

Internal Read:
 $gatherTry_p$, $p \in \mathcal{P}$
 $gatherDone_p$, $p \in \mathcal{P}$

Internal Write:
 $setNext_p$, $p \in \mathcal{P}$
 $done_p$, $p \in \mathcal{P}$

State:

$STATUS_p \in \{comp_next, set_next, gather_try, gather_done, check, do, done, end, stop\}$,
initially $STATUS_p = comp_next$
 $FREE_p, DONE_p, TRY_p \subseteq \mathcal{J}$, initially $FREE_p = \mathcal{J}$ and $DONE_p = TRY_p = \emptyset$
 $POS_p = \{POS_p(1), \dots, POS_p(m)\}$, where $POS_p(i) \in \{1, \dots, n\}$, initially $POS_p(i) = 1$
 $NEXT_p \in \{1, \dots, n+1\}$, initially undefined
 $TMP_p \in \{1, \dots, n\}$, initially undefined
 $Q_p \in \{1, \dots, m\}$, initially 1

Transitions of process p :

Input $stop_p$ Effect: $STATUS_p \leftarrow stop$	Internal Read $gatherDone_p$ Precondition: $STATUS_p = gather_done$ Effect: if $Q_p \neq p$ then $TMP_p \leftarrow done_{Q_p, POS_p(Q_p)}$ if $POS_p(Q_p) \leq n$ AND $TMP_p > 0$ then $DONE_p \leftarrow DONE_p \cup \{TMP_p\}$ $FREE_p \leftarrow FREE_p \setminus \{TMP_p\}$ $POS_p(Q_p) = POS_p(Q_p) + 1$ else $Q_p \leftarrow Q_p + 1$ end else $Q_p \leftarrow Q_p + 1$ end if $Q_p > m$ then $Q_p \leftarrow 1$ $STATUS_p \leftarrow check$ end	Internal Read $gatherTry_p$ Precondition: $STATUS_p = gather_try$ Effect: if $Q_p \neq p$ then $TMP_p \leftarrow next_{Q_p}$ if $TMP_p \leq n$ then $TRY_p \leftarrow TRY_p \cup \{TMP_p\}$ end end if $Q_p + 1 \leq m$ then $Q_p \leftarrow Q_p + 1$ else $Q_p \leftarrow 1$ $STATUS_p \leftarrow gather_done$ end
Internal $compNext_p$ Precondition: $STATUS_p = comp_next$ Effect: if $ FREE_p \setminus TRY_p \geq \beta$ then $TMP_p \leftarrow \frac{ FREE_p - (m-1)}{m}$ if $TMP_p \geq 1$ then $TMP_p \leftarrow \lfloor (p-1) \cdot TMP_p \rfloor + 1$ $NEXT_p \leftarrow rank(FREE_p, TRY_p, TMP_p)$ else $NEXT_p \leftarrow rank(FREE_p, TRY_p, p)$ end $Q_p \leftarrow 1$ $TRY_p \leftarrow \emptyset$ $STATUS_p \leftarrow set_next$ else $STATUS_p \leftarrow end$ end	Internal Write $done_p$ Precondition: $STATUS_p = done$ Effect: $done_{p, POS_p(p)} \leftarrow NEXT_p$ $DONE_p \leftarrow DONE_p \cup \{NEXT_p\}$ $FREE_p \leftarrow FREE_p \setminus \{NEXT_p\}$ $POS_p(p) \leftarrow POS_p(p) + 1$ $STATUS_p \leftarrow comp_next$	Internal Write $setNext_p$ Precondition: $STATUS_p = set_next$ Effect: $next_p \leftarrow NEXT_p$ $STATUS_p \leftarrow gather_try$
Internal $check_p$ Precondition: $STATUS_p = check$ Effect: if $NEXT_p \notin TRY_p$ AND $NEXT_p \notin DONE_p$ then $STATUS_p \leftarrow do$ else $STATUS_p \leftarrow comp_next$ end	Output $do_{p,j}$ Precondition: $STATUS_p = do$ $NEXT_p = j$ Effect: $STATUS_p \leftarrow done$	

Figure 1: Algorithm KK_β : Shared Variables, Signature, States and Transitions

$done$ is an $m * n$ matrix. In line q of the matrix, process q announces the jobs it has performed. Each cell of line q contains the identifier of exactly one job that has been performed by process q . Only process q writes in the cells of line q but any process may read them. Moreover, process q updates line q by adding entries at the end of it.

Internal Variables of process p . The variable $STATUS_p \in \{comp_next, set_next, gather_try, gather_done, check, do, done, end, stop\}$ records the status of process p and defines its next action as follows: *comp_next* - process p is ready to compute the next job to perform (this is the initial status of p), *set_next* - p computed the next job to perform and is ready to announce it, *gather_try* - p reads the array *next* in shared memory in order to compute the TRY set, *gather_done* - p reads the matrix *done* in

shared memory in order to update the DONE and FREE sets, *check* - p has to check whether it is safe to perform its current job, *do* - p can safely perform its current job, *done* - p performed its current job, *end* - p terminated, *stop* - p crashed.

$\text{FREE}_p, \text{DONE}_p, \text{TRY}_p \subseteq \mathcal{J}$ are three sets that are used by process p in order to compute the next job to perform and whether it is safe to perform it. We use some tree structure like *red-black tree* or some variant of *B-tree* [3, 11] for the sets FREE_p , DONE_p and TRY_p , in order to be able to add, remove and search elements in them in $O(\log n)$. FREE_p , is initially set to \mathcal{J} and contains an estimate of the jobs that are still available. DONE_p is initially empty and contains an estimate of the jobs that have been performed. No job is removed from DONE_p or added to FREE_p during the execution of algorithm KK_β . TRY_p is initially empty and contains an estimate of the jobs that other processes are about to perform. It holds that $|\text{TRY}_p| < m$, since there are $m - 1$ processes apart from process p that may be attempting to perform a job.

POS_p is an array of m elements. Position $\text{POS}_p(q)$ of the array contains a pointer in the line q of the shared matrix *done*. $\text{POS}_p(q)$ is the element of line q that process p will read from. In the special case where $q = p$, $\text{POS}_p(p)$ is the element of line p that process p will write into after performing a new job. The elements of the shared matrix *done* are read when process p is updating the DONE_p set.

NEXT_p contains the job process p is attempting to perform.

TMP_p is a temporary storage for values read from the shared memory.

$Q_p \in \{1, \dots, m\}$ is used as indexing for looping through process identifiers.

Actions of process p . We visit them one by one below.

compNext_p: Process p computes the set $\text{FREE}_p \setminus \text{TRY}_p$ and if it has more or equal elements to β , were β is the termination parameter of the algorithm, process p computes its next candidate job, by splitting the $\text{FREE}_p \setminus \text{TRY}_p$ set in m parts and picking the first element of the p -th part. In order to do that it uses the function $\text{rank}(\text{SET}_1, \text{SET}_2, i)$, which returns the element of set $\text{SET}_1 \setminus \text{SET}_2$ with rank i . Finally process p sets the TRY_p set to the empty set, the Q_p internal variable to 1 and its status to *set_next* in order to update the shared memory with its new candidate job. If the $\text{FREE}_p \setminus \text{TRY}_p$ set has less than β elements process p terminates.

setNext_p: Process p announces its new candidate job by writing the contents of its NEXT_p internal variable in the p -th position of the *next* array. Remember that the *next* array is stored in the shared memory. Process p changes its status to *gather_try*, in order to start collecting the TRY_p set from the *next* array.

gatherTry_p: With this action process p implements a loop, which reads from the shared memory all the positions of the array *next* and updates the TRY_p set. In each execution of the action, process p checks if Q_p is equal with p . If it is not equal, p reads the Q_p -th position of the array *next*, checks if the value read is less than $n + 1$ and if it is, adds the value it read in the TRY_p set. If Q_p is equal with p , p just skips the step described above. Then p checks if the value of $Q_p + 1$ is less than $m + 1$. If it is, then p increases Q_p by 1 and leaves its status *gather_try*, otherwise p has finished updating the TRY_p set and thus sets Q_p to 1 and changes its status to *gather_done*, in order to update the DONE_p and FREE_p sets from the contents of the *done* matrix.

gatherDone_p: With this action process p implements a loop, which updates the DONE_p and FREE_p sets with values read from the matrix *done*, which is stored in shared memory. In each execution of the action, process p checks if Q_p is equal with p . If it is not equal, p uses the internal variable $\text{POS}_p(Q_p)$, in order to read fresh values from the line Q_p of the *done* matrix. In detail, p reads the shared variable $\text{done}_{Q_p, \text{POS}_p(Q_p)}$, checks if $\text{POS}_p(Q_p)$ is less than $n + 1$ and if the value read is greater than 0. If both conditions hold, p adds the value read at the DONE_p set, removes the value read from the FREE_p set and increases $\text{POS}_p(Q_p)$ by one. Otherwise, it means that either process Q_p has terminated (by performing all the n jobs) or the line Q_p does not contain any new completed jobs. In either case p increases the value of Q_p by 1. The value of Q_p is increased by 1 also if Q_p was equal with p . Finally p checks whether Q_p is greater than m ; if it is, p has completed the loop and thus changes its status to *check*.

$check_p$: Process p checks if it is safe to perform its current job. This is done by checking if $NEXT_p$ belongs to the set TRY_p or to the set $DONE_p$. If it does not, then it is safe to perform the job $NEXT_p$ and p changes its status to do . Otherwise it is not safe, and thus p changes its status to $comp_next$, in order to find a new job that may be safe to perform.

$do_{p,j}$: Process p performs job j . Note that $NEXT_p = j$ is part of the preconditions for the action to be enabled in a state. Then p changes its status to $done$.

$done_p$: Process p writes in the $done_{p, POS_p(p)}$ position of the shared memory the value of $NEXT_p$, letting other processes know that it performed job $NEXT_p$. Also p adds $NEXT_p$ to its $DONE_p$ set, removes $NEXT_p$ from its $FREE_p$ set, increases $POS_p(p)$ by 1 and changes its status to $comp_next$.

$stop_p$: Process p crashes by setting its status to $stop$.

4 Correctness and Effectiveness Analysis

Next we begin the analysis of algorithm KK_β , by proving that KK_β solves the at-most-once problem. That is, there exists no execution of KK_β in which 2 distinct actions $do_{p,i}$ and $do_{q,i}$ appear for some $i \in \mathcal{J}$ and $p, q \in \mathcal{P}$. In the proofs, for a state s and a process p we denote by $s.FREE_p$, $s.DONE_p$, $s.TRY_p$, the values of the internal variables $FREE$, $DONE$ and TRY of process p in state s . Moreover with $s.next$, and $s.done$ we denote the contents of the array $next$ and the matrix $done$ in state s . Remember that $next$ and $done$, are stored in shared memory.

Lemma 4.1 *There exists no execution α of algorithm KK_β , such that $\exists i \in \mathcal{J}$ and $\exists p, q \in \mathcal{P}$ for which $do_{p,i}, do_{q,i} \in \alpha$.*

Proof. Let us for the sake of contradiction assume that there exists an execution $\alpha \in execs(KK_\beta)$ and $i \in \mathcal{J}$ and $p, q \in \mathcal{P}$ such that $do_{p,i}, do_{q,i} \in \alpha$. We examine two cases.

Case 1 $p = q$: Let states $s_1, s'_1, s_2, s'_2 \in \alpha$, such that the transitions $(s_1, do_{p,i}, s'_1), (s_2, do_{p,i}, s'_2) \in \alpha$ and without loss of generality assume $s'_1 \leq s_2$ in α . From Figure 1 we have that $s'_1.NEXT_p = i, s'_1.STATUS_p = done$ and $s_2.NEXT_p = i, s_2.STATUS_p = do$. From algorithm KK_β , state s_2 must be preceded in α by transition $(s_3, check_p, s'_3)$, such that $s_3.NEXT_p = i$ and $s'_3.NEXT_p = i, s'_3.STATUS_p = do$, where s'_1 precedes s_3 in α . Finally s_3 must be preceded in α by transition $(s_4, done_p, s'_4)$, where s'_1 precedes s_4 , such that $s_4.NEXT_p = i$ and $i \in s'_4.DONE_p$. Since s'_4 precedes s_3 and during the execution of KK_β no elements are removed from $DONE_p$, we have that $i \in s_3.DONE_p$. This is a contradiction, since the transition $(\{NEXT_p = i, i \in DONE_p\}, check_p, \{NEXT_p = i, STATUS_p = do\}) \notin trans(KK_\beta)$.

Case 2 $p \neq q$: Given the transition $(s_1, do_{p,i}, s'_1)$ in α , we deduce from Figure 1 that there exist in α transitions $(s_2, setNext_p, s'_2), (s_3, gatherTry_p, s'_3), (s_4, check_p, s'_4)$, where $s'_2.next_p = s'_2.NEXT_p = i, s_3.next_p = s_3.NEXT_p = i, s_3.Q_p = q, s_4.NEXT_p = i, s'_4.NEXT_p = i, s'_4.STATUS_p = do$, such that $s_2 < s_3 < s_4 < s_1$ and there exists no action $\pi = compNext_p$ in α between states s_2 and s'_1 . This essentially means that in the execution fragment $\alpha' \in \alpha$ starting from state s_2 and ending with s'_1 there exists only a single $check_p$ action - the one in transition $(s_4, check_p, s'_4)$ - that leads in the performance of job i . Similarly for transition $(t_1, do_{q,i}, t'_1)$ there exist in α transitions $(t_2, setNext_q, t'_2), (t_3, gatherTry_q, t'_3), (t_4, check_q, t'_4)$, where $t'_2.next_q = t'_2.NEXT_q = i, t_3.next_q = t_3.NEXT_q = i, t_3.Q_q = p, t_4.NEXT_q = i, t'_4.NEXT_q = i, t'_4.STATUS_q = do$, such that $t_2 < t_3 < t_4 < t_1$ and there is no action $\pi' = compNext_q$ occurring in α between states t_2 and t'_1 .

In the execution α , either state $s_2 < t_3$ or $t_3 < s_2$ which implies $t_2 < s_3$. We will show that if $s_2 < t_3$ then $\text{do}_{q,i}$ cannot take place, leading to a contradiction. The case were $t_2 < s_3$ is symmetric and will be omitted. So let us assume that s_2 precedes t_3 in α . We have two cases, either $t_3.\text{next}_p = i$ or $t_3.\text{next}_p \neq i$. In the first case $i \in t_3.\text{TRY}_q$. From Figure 1 the only action in which entries are removed from the TRY_q set, is the compNext_q where the TRY_q set is reset to \emptyset . This means that $i \in t_4.\text{TRY}_q$ since $\nexists \pi' = \text{compNext}_q \in \alpha$, such that $t_2 < \pi' < t_1$. This is a contradiction since $(t_4, \text{check}_q, t_4') \notin \text{trans}(\text{KK}_\beta)$, if $i \in t_4.\text{TRY}_q$, $t_4.\text{NEXT}_q = i$ and $t_4'.\text{STATUS}_q = \text{do}$.

If $t_3.\text{next}_p \neq i$, since $(s_2, \text{setNext}_p, s_2') \in \alpha$ and $s_2' < t_3$ there exists action $\pi_1 = \text{setNext}_p \in \alpha$, such that $s_2' < \pi_1 < t_3$. Moreover from Figure 1, there exists action $\pi_2 = \text{compNext}_p$ in α , such that $s_2' < \pi_2 < \pi_1$. Since $\nexists \pi = \text{compNext}_p \in \alpha$, such that $s_2 < \pi < s_1'$, it holds that $s_1' < \pi_2 < \pi_1 < t_3$. Furthermore, from Figure 1 there exists transition $(s_5, \text{done}_p, s_5')$ in α and $j \in \{1, \dots, n\}$, such that $s_5.\text{POS}_p(p) = j$, $s_5.\text{done}_{p,j} = 0$, $s_5.\text{NEXT}_p = i$, $s_5'.\text{done}_{p,j} = i$ and $s_1' < s_5' < \pi_2 < t_3$. It must be the case that $i \notin t_2.\text{DONE}_q$, since $t_2.\text{NEXT}_q = i$. From that and from Figure 1 we have that there exists transition $(t_6, \text{gatherDone}_q, t_6')$ in α , such that $t_6.Q_q = p$, $t_6.\text{POS}_q(p) = j$ and $t_3 < t_6 < t_4$. Since $s_5' < t_3$ and $\text{done}_{p,j}$ from Figure 1 cannot be changed again in execution α , we have that $t_6.\text{done}_{p,j} = i$ and as a result $i \in t_6'.\text{DONE}_q$. Moreover during the execution of algorithm KK_β entries in set DONE_q are only added and never removed, thus we have that $i \in t_4.\text{DONE}_q$. This is a contradiction since $(t_4, \text{check}_q, t_4') \notin \text{trans}(\text{KK}_\beta)$, if $i \in t_4.\text{DONE}_q$, $t_4.\text{NEXT}_q = i$ and $t_4'.\text{STATUS}_q = \text{do}$. This completes the proof. \square

Next we examine the effectiveness of the algorithm.

Lemma 4.2 *For any $\beta \geq m$, $f \leq m - 1$ and for any finite execution $\alpha \in \text{execs}(\text{KK}_\beta)$ with $\text{Do}(\alpha) \leq n - (\beta + m - 1)$, there exists a (non-empty) execution fragment α' such that $\alpha \cdot \alpha' \in \text{execs}(\text{KK}_\beta)$.*

Proof. From the algorithm KK_β , we have that for any process p and any state $s \in \alpha$, $|s.\text{FREE}_p| \geq n - \text{Do}(\alpha)$ and $|s.\text{TRY}_p| \leq m - 1$. The first inequality holds since the $s.\text{FREE}_p$ set is estimated by p by examining the *done* matrix which is stored in shared memory. From Figure 1 a job j is only inserted in line q of the matrix *done*, if a $\text{do}_{q,j}$ action has already been performed by process q . The second inequality is obvious. Thus we have that $\forall p \in \mathcal{P}$ and $\forall s \in \alpha$, $|s.\text{FREE}_p \setminus s.\text{TRY}_p| \geq n - (\text{Do}(\alpha) + m - 1)$. If $\text{Do}(\alpha) \leq n - (\beta + m - 1)$, $\forall p \in \mathcal{P}$ and $\forall s \in \alpha$ we have that $|s.\text{FREE}_p \setminus s.\text{TRY}_p| \geq \beta$. Since there can be $f \leq m - 1$ failed processes in our system, at the final state s' of α there exists at least one process $p \in \mathcal{P}$ that has not failed. This process has not terminated, since from Figure 1 a process p can only terminate if in the enabling state s of action compNext_p , $|s.\text{FREE}_p \setminus s.\text{TRY}_p| < \beta$. This process can continue executing steps and thus there exists (non-empty) execution fragment α' such that $\alpha \cdot \alpha' \in \text{execs}(\text{KK}_\beta)$. \square

This means that if the KK_β algorithm has effectiveness less then or equal to $n - (\beta + m - 1)$, there should be some infinite fair execution α of the algorithm with $\text{Do}(\alpha) \leq n - (\beta + m - 1)$ (since no finite execution of algorithm could terminate). Next we prove that the algorithm KK_β is wait-free (the algorithm has no infinite fair executions) and thus there exists no such execution $\alpha \in \text{execs}(\text{KK}_\beta)$.

Lemma 4.3 *For any $\beta \geq m$, $f \leq m - 1$ there exists no infinite fair execution $\alpha \in \text{execs}(\text{KK}_\beta)$.*

Proof. We will prove this by contradiction. Let $\beta \geq m$ and $\alpha \in \text{execs}(\text{KK}_\beta)$ an infinite fair execution with $f \leq m - 1$ failures, and let $\text{Do}(\alpha)$ be the jobs executed by execution α according to Definition 2.1. Since $\alpha \in \text{execs}(\text{KK}_\beta)$ and from Lemma 4.1 KK_β solves the at-most-once problem, $\text{Do}(\alpha)$ is finite. Clearly there exists at least one process in α that has not crashed and does not terminate (some process need to take steps in α in order for it to be infinite). Since $\text{Do}(\alpha)$ and f are finite, there exists a state s_0 in α such that after s_0 no process crashes, no process terminates, no do action takes place in α and no process adds new

entries in the *done* matrix in shared memory. The later holds since the execution is infinite and fair, the $Do(\alpha)$ is also finite, consequently any non failed process q that has not terminated will eventually update the q line of the *done* matrix to be in agreement with the $do_{q,*}$ actions it has performed. Moreover any process q that has terminated, has already updated the q line of *done* matrix with the latest *do* action it performed, before it terminated, since in order to terminate it must have reached a *compNext* action that has set its status to *end*.

We define the following sets of processes and jobs according to state s_0 . \mathcal{J}_α are jobs that have been performed in α according to Definition 2.1. \mathcal{P}_α are processes that do not crash and do not terminate in α . By the way we defined state s_0 only processes in \mathcal{P}_α take steps in α after state s_0 . $STUCK_\alpha = \{i \in \mathcal{J} \setminus \mathcal{J}_\alpha \mid \exists \text{ failed process } p : s_0.next_p = i\}$, i.e., $STUCK_\alpha$ expresses the set of jobs that are held by failed processes. $DONE_\alpha = \{i \in \mathcal{J}_\alpha \mid \exists p \in \mathcal{P} \text{ and } j \in \{1, \dots, n\} : s_0.done_p(j) = i\}$, i.e., $DONE_\alpha$ expresses the set of jobs that have been performed before state s_0 and the processes that performed them managed to update the shared memory. Finally we define $POOL_\alpha = \mathcal{J} \setminus (\mathcal{J}_\alpha \cup STUCK_\alpha)$. After state s_0 , all processes in \mathcal{P}_α will keep executing. This means that whenever such process $p \in \mathcal{P}_\alpha$ takes action $compNext_p$ in α , the first if statement is true. Specifically it holds that for $\forall p \in \mathcal{P}_\alpha$ and for all the enabling states $s \geq s_0$ of actions $compNext_p$ in α , $|FREE_p \setminus TRY_p| \geq \beta$.

From Figure 1, we have that for any $p \in \mathcal{P}_\alpha$, $\exists s_p$ after state s_0 in α such that \forall states $s \geq s_p$, $s.DONE_p = DONE_\alpha$, $s.FREE_p = \mathcal{J} \setminus DONE_\alpha$ and $s.FREE_p \setminus s.TRY_p \subseteq POOL_\alpha$. Let $s'_0 = \max_{p \in \mathcal{P}_\alpha} [s_p]$. From the above we have: $|\mathcal{J} \setminus DONE_\alpha| \geq \beta \geq m$ and $|POOL_\alpha| \geq \beta \geq m$, since $\forall p \in \mathcal{P}_\alpha$ we have that for all the enabling states $s \geq s'_0$ of actions $compNext_p$ in α $|FREE_p \setminus TRY_p| \geq \beta$ and $\forall s' \geq s'_0$ we have that $s'.FREE_p = \mathcal{J} \setminus DONE_\alpha$ and $s'.FREE_p \setminus s'.TRY_p \subseteq POOL_\alpha$.

Let p_0 be the process with the smallest process identifier in \mathcal{P}_α . We examine 2 cases according to the size of $\mathcal{J} \setminus DONE_\alpha$.

Case A $|\mathcal{J} \setminus DONE_\alpha| \geq 2m - 1$: Let $x_0 \in POOL_\alpha$ be the job such that $[x_0]_{POOL_\alpha} = \left\lfloor (p_0 - 1) \cdot \frac{|\mathcal{J} \setminus DONE_\alpha| - (m-1)}{m} \right\rfloor + 1$. Such x_0 exists since $\forall p \in \mathcal{P}_\alpha$ and $\forall s \geq s'_0$ it holds $s.FREE_p \setminus s.TRY_p \subseteq POOL_\alpha$, $s.FREE_p = \mathcal{J} \setminus DONE_\alpha$ from which we have that $|POOL_\alpha| \geq |\mathcal{J} \setminus DONE_\alpha| - |s.TRY_p| \geq |\mathcal{J} \setminus DONE_\alpha| - (m-1) \geq m$.

It follows that any $p \in \mathcal{P}_\alpha$ that executes action $compNext_p$ after state s'_0 , will have its $NEXT_p$ variable pointing in a job x with $[x]_{POOL_\alpha} \geq \left\lfloor (p-1) \cdot \frac{|\mathcal{J} \setminus DONE_\alpha| - (m-1)}{m} \right\rfloor + 1$. Thus $\forall p \in \mathcal{P}_\alpha$, $\exists s'_p \geq s'_0$ in α such that \forall states $s \geq s'_p$, $[s.next_p]_{POOL_\alpha} \geq \left\lfloor (p-1) \cdot \frac{|\mathcal{J} \setminus DONE_\alpha| - (m-1)}{m} \right\rfloor + 1$. Let $s''_0 = \max_{p \in \mathcal{P}_\alpha} [s'_p]$, we have to study 2 cases for p_0 :

Case A.1) After s''_0 , process p_0 executes action $compNext_{p_0}$ and the transition leads in state $s_1 > s''_0$ such that $s_1.NEXT_{p_0} = x_0$. Since $[x_0]_{POOL_\alpha} = \left\lfloor (p_0 - 1) \cdot \frac{|\mathcal{J} \setminus DONE_\alpha| - (m-1)}{m} \right\rfloor + 1$ and $p_0 = \min_{p \in \mathcal{P}_\alpha} [p]$, from the previous discussion we have that $\forall s \geq s_1$ and $\forall p \in \mathcal{P} \setminus \{p_0\}$, $s.next_p \neq x_0$. Thus when p_0 executes action $check_p$ of Figure 1 for the first time after state s_1 , the condition will be true, so in some subsequent transition p_0 will have to execute action do_{p_0, x_0} , performing job x_0 , which is a contradiction, since after state s_0 no jobs are executed.

Case A.2) After s''_0 , process p_0 executes action $compNext_{p_0}$ and the transition leads in state $s_1 > s''_0$ such that $s_1.NEXT_{p_0} > x_0$. Since $p_0 = \min_{p \in \mathcal{P}_\alpha} [p]$, it holds that $\forall x \in POOL_\alpha$ such that $[x]_{POOL_\alpha} \leq \left\lfloor (p_0 - 1) \cdot \frac{|\mathcal{J} \setminus DONE_\alpha| - (m-1)}{m} \right\rfloor + 1$, $\nexists p \in \mathcal{P}$ such that $s_1.next_p = x$. Let the transition $(s_2, compNext_{p_0}, s'_2) \in \alpha$, where $s_2 > s_1$, be the first time that action $compNext_{p_0}$ is executed after state s_1 . We have that $\forall x \in POOL_\alpha$ such that $[x]_{POOL_\alpha} \leq \left\lfloor (p_0 - 1) \cdot \frac{|\mathcal{J} \setminus DONE_\alpha| - (m-1)}{m} \right\rfloor + 1$, $x \notin s_2.DONE_{p_0} \cup s_2.TRY_{p_0}$, since from the discussion above we have that $\forall s \geq s_1$ and $\forall p \in \mathcal{P}_\alpha \setminus \{p_0\}$, $[s.next_p]_{POOL_\alpha} \geq \left\lfloor (p-1) \cdot \frac{|\mathcal{J} \setminus DONE_\alpha| - (m-1)}{m} \right\rfloor + 1$. Thus $[x_0]_{s_2.FREE_{p_0} \setminus s_2.TRY_{p_0}} = [x_0]_{POOL_\alpha} =$

$\left\lfloor (p_0 - 1) \cdot \frac{|\mathcal{J} \setminus \text{DONE}_\alpha| - (m-1)}{m} \right\rfloor + 1$. As a result, $s'_2.\text{NEXT}_{p_0} = x_0$. With similar arguments like in case A.1, we can see that job x_0 will be performed by process p_0 , which is a contradiction, since after state s_0 no jobs are executed.

Case B $|\mathcal{J} \setminus \text{DONE}_\alpha| < 2m - 1$: Let $x_0 \in \text{POOL}_\alpha$ be the job such that $[x_0]_{\text{POOL}_\alpha} = p_0$. Such x_0 exists since $\beta \geq m$ and $\text{POOL}_\alpha \geq \beta$. It follows that any $p \in \mathcal{P}_\alpha$ that executes action compNext_p after state s'_0 , will have its NEXT_p variable pointing in a job x with $[x]_{\text{POOL}_\alpha} \geq p$. Thus $\forall p \in \mathcal{P}_\alpha, \exists s'_p \geq s'_0$ in α such that \forall states $s \geq s'_p, [s.\text{next}_p]_{\text{POOL}_\alpha} \geq p$. Let $s''_0 = \max_{p \in \mathcal{P}_\alpha} [s'_p]$, we have to study 2 cases for p_0 :

Case B.1) After s''_0 , process p_0 executes action compNext_{p_0} and the transition leads in state $s_1 > s''_0$ such that $s_1.\text{NEXT}_{p_0} = x_0$. Since $[x_0]_{\text{POOL}_\alpha} = p_0$ and $p_0 = \min_{p \in \mathcal{P}_\alpha} [p]$, from the previous discussion we have that $\forall s \geq s_1$ and $\forall p \in \mathcal{P} \setminus \{p_0\}, s.\text{next}_p \neq x_0$. Thus when p_0 executes action check_p of Figure 1 for the first time after state s_1 , the condition will be true, so in some subsequent transition p_0 will have to execute action do_{p_0, x_0} , performing job x_0 , which is a contradiction, since after state s_0 no jobs are executed.

Case B.2) After s''_0 , process p_0 executes action compNext_{p_0} and the transition leads in state $s_1 > s''_0$ such that $s_1.\text{NEXT}_{p_0} > x_0$. Since $p_0 = \min_{p \in \mathcal{P}_\alpha} [p]$, it holds that $\forall x \in \text{POOL}_\alpha$ such that $[x]_{\text{POOL}_\alpha} \leq p_0, \nexists p \in \mathcal{P}$ such that $s_1.\text{next}_p = x$. Let the transition $(s_2, \text{compNext}_{p_0}, s'_2) \in \alpha$, where $s_2 > s_1$, be the first time that action compNext_{p_0} is executed after state s_1 . We have that $\forall x \in \text{POOL}_\alpha$ such that $[x]_{\text{POOL}_\alpha} \leq p_0, x \notin s_2.\text{DONE}_{p_0} \cup s_2.\text{TRY}_{p_0}$, since from the discussion above we have that $\forall s \geq s_1$ and $\forall p \in \mathcal{P}_\alpha \setminus \{p_0\}, [s.\text{next}_p]_{\text{POOL}_\alpha} \geq p$. Thus $[x_0]_{s_2.\text{FREE}_{p_0} \setminus s_2.\text{TRY}_{p_0}} = [x_0]_{\text{POOL}_\alpha} = p_0$. As a result, $s'_2.\text{NEXT}_{p_0} = x_0$. With similar arguments like in case B.1, we can see that job x_0 will be performed by process p_0 , which is a contradiction, since after state s_0 no jobs are executed. \square

Using the last two lemmas we can find the effectiveness of algorithm KK_β .

Theorem 4.4 For any $\beta \geq m, f \leq m - 1$ algorithm KK_β has effectiveness $E_{\text{KK}_\beta}(n, m, f) = n - (\beta + m - 2)$.

Proof. From Lemma 4.2 we have that any finite execution $\alpha \in \text{execs}(\text{KK}_\beta)$ with $\text{Do}(\alpha) \leq n - (\beta + m - 1)$ can be extended, essentially proving that in such an execution no process has terminated. Moreover from Lemma 4.3 we have that KK_β is wait free, and thus there exists no infinite fair execution $\alpha \in \text{execs}(\text{KK}_\beta)$, such that $\text{Do}(\alpha) \leq n - (\beta + m - 1)$. Since finite fair executions are executions where all non-failed processes have terminated, from the above we have that $E_{\text{KK}_\beta}(n, m, f) \geq n - (\beta + m - 2)$.

If all processes but the process with id m fail in an execution α in such a way that $\mathcal{J}_\alpha \cap \text{STUCK}_\alpha = \emptyset$ and $|\text{STUCK}_\alpha| = m - 1$ (where STUCK_α is defined as in the proof of lemma 4.3), then there exists adversarial strategy, that can result in $\beta + m - 2$ jobs not having been performed when process m terminates. Such an execution will be a finite fair execution where $n - (\beta + m - 2)$ jobs are performed. From this and the previous claims we have that $E_{\text{KK}_\beta}(n, m, f) = n - (\beta + m - 2)$. \square

5 Work Complexity Analysis

In this section we are going to prove that for $\beta \geq 3m^2$ algorithm KK_β has work complexity $O(nm \log n \log m)$.

The main idea of the proof, is to demonstrate that under the assumption $\beta \geq 3m^2$, process collisions on a job cannot accrue without making progress in the algorithm. In order to prove that, we first demonstrate that if two different processes p, q set their $\text{NEXT}_p, \text{NEXT}_q$ internal variables to the same job i in some compNext actions, then at the enabling states of those actions the DONE_p and DONE_q sets of the processes, have at least $|q - p|m$ different elements, given that $\beta \geq 3m^2$. Next we prove that if two processes p, q collide three consecutive times, while trying to perform some jobs, then the size of the set $\text{DONE}_p \cup \text{DONE}_q$ that

processes p and q know has increased by at least $|q - p|m$ elements. This essentially tells us that every three collisions between the same two processes a significant number of jobs has been performed, and thus enough progress has been made. In order to prove the above statement, we need to formally define what we mean by collision, and tie such a collision with some specific state, so that we have a fixed “point” in the execution for which to reason. Finally we use the argument about the progress made if three consecutive collisions happen between two processes p, q , in order to prove that a process p cannot collide with a process q more than $2 \left\lceil \frac{n}{m|q-p|} \right\rceil$ times in any execution. This is proved by contradiction, proving that if process p collides with process q more than $2 \left\lceil \frac{n}{m|q-p|} \right\rceil$ times, there exist states for which the set $|\text{DONE}_p \cup \text{DONE}_q|$ has more than n elements which is impossible. The last statement is used in order to prove the main theorem on the work complexity of algorithm KK_β for $\beta \geq 3m^2$. We obtain the main theorem on the work complexity by counting the total number of collisions and the cost of each collision.

We start by proving that if two processes p, q decide, with some compNext actions, to perform the same job i , then their DONE sets at the enabling states of those compNext actions, differ in at-least $|q - p|m$ elements.

Lemma 5.1 *If $\beta \geq 3m^2$ and in an execution $\alpha \in \text{execs}(\text{KK}_\beta)$ there exist states s_1, t_1 and processes $p, q \in \mathcal{P}$ with $p < q$ such that $s_1.\text{NEXT}_p = t_1.\text{NEXT}_q = i \in \mathcal{J}$, then there exist transitions $(s_2, \text{compNext}_p, s'_2)$, $(t_2, \text{compNext}_q, t'_2)$, where $s'_2.\text{NEXT}_p = t'_2.\text{NEXT}_q = i$, $s'_2.\text{STATUS}_p = t'_2.\text{STATUS}_q = \text{set_next}$, such that there exist no action $\pi_1 = \text{compNext}_p$ with $s'_2 < \pi_1 < s_1$ and no action $\pi_2 = \text{compNext}_q$ with $t'_2 < \pi_2 < t_1$ and $|s_2.\text{DONE}_p \cap t_2.\text{DONE}_q| > (q - p)m$ or $|s_2.\text{DONE}_p \cap t_2.\text{DONE}_q| > (q - p)m$*

Proof. We will prove this by contradiction. From algorithm KK_β there must exist transitions $(s_2, \text{compNext}_p, s'_2)$, $(t_2, \text{compNext}_q, t'_2)$ where $s'_2.\text{NEXT}_p = i$ and $t'_2.\text{NEXT}_q = i$, such that there exist no actions $\pi_1 = \text{compNext}_p$, $\pi_2 = \text{compNext}_q$ with $s'_2 < \pi_1 < s_1$ and $t'_2 < \pi_2 < t_1$, if there exist $s_1, t_1 \in \alpha$ and $p, q \in \mathcal{P}$ with $p < q$ such that $s_1.\text{NEXT}_p = t_1.\text{NEXT}_q = i \in \mathcal{J}$, since those are the transitions that set NEXT_p and NEXT_q to i . So in order to get a contradiction we must assume that $|s_2.\text{DONE}_p \cap t_2.\text{DONE}_q| \leq (q - p) \cdot m$ and $|s_2.\text{DONE}_p \cap t_2.\text{DONE}_q| \leq (q - p) \cdot m$.

We will prove that if this is the case $s'_2.\text{NEXT}_p \neq t'_2.\text{NEXT}_q$.

Let $A = \mathcal{J} \setminus s_2.\text{DONE}_p = s_2.\text{FREE}_p$ and $B = \mathcal{J} \setminus t_2.\text{DONE}_q = t_2.\text{FREE}_q$, thus from the contradiction assumption we have that: $|\overline{A} \cap B| \leq (q - p) \cdot m$ and $|A \cap \overline{B}| \leq (q - p) \cdot m$.

It could either be that $|A| < |B|$ or $|A| \geq |B|$.

Case 1 $|A| < |B|$: From the contradiction assumption we have that $|\overline{A} \cap B| \leq (q - p) \cdot m$. Thus we have that:

$$|t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q \cap \overline{s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p}| \leq m(q - p) + m - 1 \quad (1)$$

,since $s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p$ can have up to $m - 1$ less elements than A - the elements of set $s_2.\text{TRY}_p$ - and it can be the case that $s_2.\text{TRY}_p \cap t_2.\text{TRY}_q = \emptyset$.

Moreover, since $s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p \subseteq A$ and $|s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p| \geq \beta \geq 3m^2$, $|A| \geq 3m^2$. Similarly $|B| \geq 3m^2$. We have:

$$(q - 1) \frac{|B|}{m} = (p - 1) \frac{|B|}{m} + (q - p) \frac{|B|}{m} > (p - 1) \frac{|A|}{m} + (q - p) \frac{|B|}{m} \Rightarrow \quad (2)$$

$$\Rightarrow (q - 1) \frac{|B|}{m} > (p - 1) \frac{|A|}{m} + 3m(q - p) \Rightarrow \quad (3)$$

$$\Rightarrow \left\lceil (q - 1) \frac{|B| - (m - 1)}{m} \right\rceil + 1 \geq \left\lceil (p - 1) \frac{|A| - (m - 1)}{m} \right\rceil + 1 + 3m(q - p) \quad (4)$$

Since $s'_2.\text{NEXT}_p = t'_2.\text{NEXT}_q = i$, it must be the case that $[i]_{s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p} = \left\lfloor (p-1) \frac{|A|-(m-1)}{m} \right\rfloor + 1$ and $[i]_{t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q} = \left\lfloor (q-1) \frac{|B|-(m-1)}{m} \right\rfloor + 1$. Equation 4 gives that $[i]_{t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q} \geq [i]_{s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p} + 3m(q-p)$. This means that $t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q$ must have at least $3m(q-p)$ more elements with rank less than the rank of i , than set $s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p$ does. This is a contradiction since from 1 we have that $|t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q \cap s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p| \leq m(q-p) + m - 1$.

Case 2 $|B| \leq |A|$: We have that $|\overline{A} \cap B| \leq (q-p) \cdot m$ and $|A \cap \overline{B}| \leq (q-p) \cdot m$ from the contradiction assumption. Thus we have that:

$$|t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q \cap \overline{s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p}| \leq m(q-p) + m - 1 \quad (5)$$

,since $s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p$ can have up to $m-1$ less elements than A - the elements of set $s_2.\text{TRY}_p$ - and it can be the case that $s_2.\text{TRY}_p \cap t_2.\text{TRY}_p = \emptyset$.

From the contradiction assumption and the case 2 assumption we have that $|B| \leq |A| \leq |B| + (q-p)m$. Moreover $|A| \geq \beta \geq 3m^2$ and $|B| \geq \beta \geq 3m^2$. We have:

$$\begin{aligned} (q-1) \frac{|B| + (q-p)m}{m} &= (p-1) \frac{|B| + (q-p)m}{m} + (q-p) \frac{|B| + (q-p)m}{m} \geq \\ &\geq (p-1) \frac{|A|}{m} + (q-p) \frac{|B| + (q-p)m}{m} \geq (p-1) \frac{|A|}{m} + 3m(q-p) + (q-p)^2 \Rightarrow \\ &\Rightarrow (q-1) \frac{|B|}{m} \geq (p-1) \frac{|A|}{m} + 3m(q-p) + (q-p)^2 - (q-1)(q-p) \Rightarrow \\ &\Rightarrow (q-1) \frac{|B|}{m} \geq (p-1) \frac{|A|}{m} + (3m-p+1)(q-p) \Rightarrow \\ &\Rightarrow (q-1) \frac{|B|}{m} \geq (p-1) \frac{|A|}{m} + 2m(q-p) \Rightarrow \\ &\Rightarrow \left\lfloor (q-1) \frac{|B|-(m-1)}{m} \right\rfloor + 1 \geq \left\lfloor (p-1) \frac{|A|-(m-1)}{m} \right\rfloor + 1 + 2m(q-p) \end{aligned} \quad (6)$$

Since $s'_2.\text{NEXT}_p = t'_2.\text{NEXT}_q = i$, it must be the case that $[i]_{s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p} = \left\lfloor (p-1) \frac{|A|-(m-1)}{m} \right\rfloor + 1$ and $[i]_{t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q} = \left\lfloor (q-1) \frac{|B|-(m-1)}{m} \right\rfloor + 1$. Equation 6 gives that $[i]_{t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q} \geq [i]_{s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p} + 2m(q-p)$. This means that $t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q$ must have at least $2m(q-p)$ more elements with rank less than the rank of i than set $s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p$. This is a contradiction since from 5 we have that $|t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q \cap s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p| \leq m(q-p) + m - 1$. \square

Next we are going to prove that if 2 processes $p, q \in \mathcal{P}$ with $p < q$ “collide” three times, their DONE sets at the third collision will contain at least $m(q-p)$ more jobs than they did at the first collision. This will allow us to find an upper bound on the collisions a process may participate in. It is possible that both processes become aware of a collision or only one of them does while the other one successfully completes the job. At the proofs that follow, for a state s in execution α we define as $s.\text{DONE}$ the following set: $s.\text{DONE} = \{i \in \mathcal{J} \mid \exists p \in \mathcal{P} \text{ and } j \in \{1, \dots, n\} : s.\text{done}_p(j) = i\}$. We also need the following definitions.

Definition 5.1 In an execution $\alpha \in \text{execs}(\text{KK}_\beta)$, we say that process p **collided** with process q in job i at state s , if (i) there exist in α transitions $(s_1, \text{compNext}_p, s'_1)$, $(t_1, \text{compNext}_q, t'_1)$ and $(s_2, \text{check}_p, s'_2)$, with $s_1 < s_2$ and $t_1 < s_2$, $s'_1.\text{NEXT}_p = t'_1.\text{NEXT}_q = s_2.\text{NEXT}_p = i$, $s'_1.\text{STATUS}_p = t'_1.\text{STATUS}_q = \text{set_next}$, $s'_2.\text{STATUS}_p = \text{comp_next}$, (ii) let α' be the execution fragment that begins with state s'_1 and ends with state s_2 , there exists no action $\pi_1 = \text{compNext}_p \in \alpha'$ and either there exists in α' transition

$(s, \text{gatherTry}_p, s')$ such that $s.Q_p = q$, $s.\text{next}_q = i$, or transition $(s, \text{gatherDone}_p, s')$ and $j \in \{1, \dots, n\}$ such that $s.Q_p = q$, $s.\text{POS}_p(q) = j$, $s.\text{done}_{q,j} = i$ and $i \notin s.\text{TRY}_p$.

According to Def. 5.1 process p collided with process q in job i at state s , if process p attempted to preform job i , but was not able to, because it detected in state s that either process q was trying to perform job i or process q has already performed job i .

Definition 5.2 In an execution $\alpha \in \text{execs}(\text{KK}_\beta)$, we say that processes p, q **collide** in job i at state s , if process p collided with process q or process q collided with process p in job i at state s , according to Definition 5.1.

Lemma 5.2 In an execution $\alpha \in \text{execs}(\text{KK}_\beta)$ for any $\beta \geq m$ if there exist processes p, q , jobs $i_1, i_2 \in \mathcal{J}$ and states $\tilde{s}_1 < \tilde{s}_2$ such that process p collided with process q in job i_1 at state \tilde{s}_1 and in job i_2 at state \tilde{s}_2 according to Definition 5.1, then there exist transitions $(s_1, \text{compNext}_p, s'_1)$, $(s_2, \text{compNext}_p, s'_2)$, $(t_1, \text{compNext}_q, t'_1)$, $(t_2, \text{compNext}_q, t'_2)$ where $s'_1.\text{NEXT}_p = t'_1.\text{NEXT}_q = i_1$, $s'_2.\text{NEXT}_p = t'_2.\text{NEXT}_q = i_2$, $s'_1.\text{STATUS}_p = s'_2.\text{STATUS}_p = t'_1.\text{STATUS}_q = t'_2.\text{STATUS}_q = \text{set_next}$ and there exists no action $\pi_1 = \text{compNext}_p$ for which $s_1 < \pi_1 < \tilde{s}_1$, $s_2 < \pi_1 < \tilde{s}_2$ such that: $s_1 < s_2$ and $t_1 < t_2$.

Proof. From Definition 5.1 we have that there exist transitions $(s_1, \text{compNext}_p, s'_1)$, $(s_2, \text{compNext}_p, s'_2)$ with $s'_1.\text{NEXT}_p = i_1$, $s'_2.\text{NEXT}_p = i_2$, $s'_1.\text{STATUS}_p = s'_2.\text{STATUS}_p = \text{set_next}$, and there exists no action $\pi_1 = \text{compNext}_p$ for which $s_1 < \pi_1 < \tilde{s}_1$ or $s_2 < \pi_1 < \tilde{s}_2$. From the later and the fact that $\tilde{s}_1 < \tilde{s}_2$, it must be the case that $s_1 < \tilde{s}_1 < s_2 < \tilde{s}_2$. Furthermore from Definition 5.1 we have that there exist transitions $(t_1, \text{compNext}_q, t'_1)$, $(t_2, \text{compNext}_q, t'_2)$ with $t'_1.\text{NEXT}_q = i_1$, $t'_2.\text{NEXT}_q = i_2$, $t'_1.\text{STATUS}_q = t'_2.\text{STATUS}_q = \text{set_next}$, such that $t'_1 < \tilde{s}_1$ and $t'_2 < \tilde{s}_2$. We can pick those transitions in α in such a way that there exists no other transition between t'_1 and \tilde{s}_1 that sets NEXT_q to i_1 and similarly there exists no other transition between t'_2 and \tilde{s}_2 that sets NEXT_q to i_2 . We need to prove now that $t_1 < t_2$. We will prove this by contradiction.

Let $t_2 < t_1$. Since $t'_1 < \tilde{s}_1$, we have that $t_2 < t_1 < t'_1 < \tilde{s}_1 < s_2 < \tilde{s}_2$. Since from Definition 5.1 either $\tilde{s}_1.\text{next}_q = i_1$ or there exists $j \in \{1, \dots, n\}$ such that $\tilde{s}_1.\text{done}_{q,j} = i_1$, it must be the case that $\tilde{s}_2.\text{STATUS}_p = \text{gather_done}$, $\tilde{s}_2.Q_p = q$ and there exists $j' \in \{1, \dots, n\}$ such that $\tilde{s}_2.\text{done}_{p,j'} = i_2$. This means that there exists transition $(t_3, \text{done}_q, t'_3)$ and $j' \in \{1, \dots, n\}$ such that $t'_3.\text{done}_{p,j'} = i_2$ and $t_2 < t'_3 < t_1 < t'_1 < \tilde{s}_1 < s_2 < \tilde{s}_2$.

If $\tilde{s}_1.\text{STATUS}_p = \text{gather_try}$ then from algorithm KK_β we have that $\tilde{s}_1.\text{DONE} \subseteq s_2.\text{DONE}_p$ and as a result $i_2 \in s_2.\text{DONE}_p$, which is a contradiction since $(s_2, \text{compNext}_p, s'_2) \notin \text{trans}(\text{KK}_\beta)$ if $i_2 \in s_2.\text{DONE}_p$ and $s'_2.\text{NEXT}_p = i_2$, $s'_2.\text{STATUS}_p = \text{set_next}$.

If $\tilde{s}_1.\text{STATUS}_p = \text{gather_done}$ then from algorithm KK_β we have that $\tilde{s}_1.Q_p = q$ and there exists $j \in \{1, \dots, n\}$ such that $\tilde{s}_1.\text{POS}_p(q) = j$ and $\tilde{s}_1.\text{done}_{q,j} = i_1$. Since $t_2 < t'_3 < t_1 < t'_1 < \tilde{s}_1 < s_2 < \tilde{s}_2$ it must be the case that $j' < j$ and as a result $i_2 \in \tilde{s}_1.\text{DONE}_p$. Clearly $\tilde{s}_1.\text{DONE}_p \subseteq s_2.\text{DONE}_p$, which is a contradiction since $(s_2, \text{compNext}_p, s'_2) \notin \text{trans}(\text{KK}_\beta)$ if $i_2 \in s_2.\text{DONE}_p$ and $s'_2.\text{NEXT}_p = i_2$, $s'_2.\text{STATUS}_p = \text{set_next}$. \square

Lemma 5.3 In an execution $\alpha \in \text{execs}(\text{KK}_\beta)$ for any $\beta \geq m$ if there exist processes p, q , jobs $i_1, i_2 \in \mathcal{J}$ and states $\tilde{s}_1 < \tilde{s}_2$ such that process p collided with process q in job i_1 at state \tilde{s}_1 and process q collided with process p in job i_2 at state \tilde{s}_2 according to Definition 5.1, then there exist transitions $(s_1, \text{compNext}_p, s'_1)$,

$(s_2, \text{compNext}_p, s'_2), (t_1, \text{compNext}_q, t'_1), (t_2, \text{compNext}_q, t'_2)$ where $s'_1.\text{NEXT}_p = t'_1.\text{NEXT}_q = i_1$, $s'_2.\text{NEXT}_p = t'_2.\text{NEXT}_q = i_2$, $s'_1.\text{STATUS}_p = s'_2.\text{STATUS}_p = t'_1.\text{STATUS}_q = t'_2.\text{STATUS}_q = \text{set_next}$ and there exists no actions $\pi_1 = \text{compNext}_p$, $\pi_2 = \text{compNext}_q$ for which $s_1 < \pi_1 < \tilde{s}_1$, $t_2 < \pi_2 < \tilde{s}_2$ such that:

$$s_1 < s_2 \text{ and } t_1 < t_2.$$

Proof. From Definition 5.1 we have that there exist transitions $(s_1, \text{compNext}_p, s'_1), (s_2, \text{compNext}_p, s'_2)$ with $s'_1.\text{NEXT}_p = i_1$, $s'_2.\text{NEXT}_p = i_2$, $s'_1.\text{STATUS}_p = s'_2.\text{STATUS}_p = \text{set_next}$, and there exists no action $\pi_1 = \text{compNext}_p$ for which $s_1 < \pi_1 < \tilde{s}_1$. Furthermore from Definition 5.1 we have that there exist transitions $(t_1, \text{compNext}_q, t'_1), (t_2, \text{compNext}_q, t'_2)$ with $t'_1.\text{NEXT}_q = i_1$, $t'_2.\text{NEXT}_q = i_2$, $t'_1.\text{STATUS}_q = t'_2.\text{STATUS}_q = \text{set_next}$, and there exists no action $\pi_2 = \text{compNext}_q$ for which $t_2 < \pi_2 < \tilde{s}_2$. From the later and the fact that $\tilde{s}_1 < \tilde{s}_2$, it must be the case that $t_1 < \tilde{s}_1 < t_2 < \tilde{s}_2$. We can pick the transitions that are enabled by states t_1 and s_2 in α in such a way that there exists no other transition between t'_1 and \tilde{s}_1 that sets NEXT_q to i_1 and similarly there exists no other transition between s'_2 and \tilde{s}_2 that sets NEXT_p to i_2 . We need to prove now that $s_1 < s_2$. We will prove this by contraction.

Let $s_2 < s_1$. From algorithm KK_β there exist transitions $(s_3, \text{setNext}_p, s'_3), (s_4, \text{done}_p, s'_4)$ and $(t_3, \text{setNext}_q, t'_3)$, where $s'_3.\text{next}_p = i_2$, $s_4.\text{next}_p = i_2$, $t'_3.\text{next}_q = i_1$ and $s_2 < s'_3 < s_4 < s_1$, $t_1 < t'_3 < t_2$. There are 2 cases, either $s'_3 < t'_3$ or $t'_3 < s'_3$.

Case 1 $s'_3 < t'_3$: We have that $s'_3 < t'_3 < t_2$ and $(t_2, \text{compNext}_q, t'_2)$, where $t'_2.\text{NEXT}_q = i_2$ and $t'_2.\text{STATUS}_q = \text{set_next}$ which means that $i_2 \notin t_2.\text{TRY}_q \cup t_2.\text{DONE}_q$. This is a contradiction since the $t_2.\text{TRY}_q$ and $t_2.\text{DONE}_q$ are computed by gatherTry_q and gatherDone_q actions that are preceded by state s'_3 . So either $i_2 \in t_2.\text{TRY}_q$ or $i_2 \in t_2.\text{DONE}_q$, since a new setNext_p action may take place only after state s_4 .

Case 2 $t'_3 < s'_3$: We have that $t'_3 < s'_3 < s_1$ and $(s_1, \text{compNext}_p, s'_1)$, where $s'_1.\text{NEXT}_p = i_1$ and $s'_1.\text{STATUS}_p = \text{set_next}$ which means that $i_1 \notin s_1.\text{TRY}_p \cup s_1.\text{DONE}_p$. This is a contradiction since the $s_1.\text{TRY}_p$ and $s_1.\text{DONE}_p$ sets are computed by gatherTry_p and gatherDone_p actions that are preceded by state t'_3 . There exists transition $(s_4, \text{gatherTry}_p, s'_4)$ in α with $s_4.Q_p = q$ such that there exists no $\pi_1 = \text{compNext}_p$ where $s'_4 < \pi_1 < s_1$. If $s_4.\text{next}_q = i_1$ we have a contradiction since $i_1 \in s_1.\text{TRY}_p$. If $s_4.\text{next}_q \neq i_1$ there exists an action $\pi_2 = \text{setNext}_q$ in α , such that $t'_3 < \pi_2 < s_4$. If this $\pi_2 = \text{setNext}_q$ is preceded by transition $(t_4, \text{done}_q, t'_4)$ with $t_4.\text{NEXT}_q = i_1$, we have a contradiction since $i_1 \in t'_4.\text{DONE}$ and $s_1.\text{DONE}_p$ is computed by gatherDone_p actions that are preceded by state t'_4 , which results in $i_1 \in s_1.\text{DONE}_p$. If there exists no such transition we have again a contradiction since state \tilde{s}_1 as defined by Definition 5.1 could not belong in α . \square

Lemma 5.4 *If $\beta \geq m$ and in an execution $\alpha \in \text{execs}(\text{KK}_\beta)$ there exist processes $p \neq q$, jobs $i_1, i_2, i_3 \in \mathcal{J}$ and states $\tilde{s}_1 < \tilde{s}_2 < \tilde{s}_3$ such that process p, q collide in job i_1 at state \tilde{s}_1 , in job i_2 at state \tilde{s}_2 and in job i_3 at state \tilde{s}_3 according to Definition 5.2, then there exist states $s_1 < s_3$ and $t_1 < t_3$ such that*

$$s_1.\text{DONE}_p \cup t_1.\text{DONE}_q \subseteq s_3.\text{DONE}_p \cap t_3.\text{DONE}_q$$

$$|s_3.\text{DONE}_p \cup t_3.\text{DONE}_q| - |s_1.\text{DONE}_p \cup t_1.\text{DONE}_q| \geq m \cdot |q - p|$$

Proof. From Definitions 5.1, 5.2 we have that there exist transitions $(s_1, \text{compNext}_p, s'_1)$, $(s_2, \text{compNext}_p, s'_2)$, $(s_3, \text{compNext}_p, s'_3)$ and $(t_1, \text{compNext}_q, t'_1)$, $(t_2, \text{compNext}_q, t'_2)$, $(t_3, \text{compNext}_q, t'_3)$, where $s'_1.\text{NEXT}_p = t'_1.\text{NEXT}_q = i_1$, $s'_2.\text{NEXT}_p = t'_2.\text{NEXT}_q = i_2$, $s'_3.\text{NEXT}_p = t'_3.\text{NEXT}_q = i_3$, $s'_1.\text{STATUS}_p = s'_2.\text{STATUS}_p = s'_3.\text{STATUS}_p = t'_1.\text{STATUS}_q = t'_2.\text{STATUS}_q = t'_3.\text{STATUS}_q = \text{set_next}$ and $s_1 < \tilde{s}_1$, $t_1 < \tilde{t}_1$, $s_2 < \tilde{s}_2$, $t_2 < \tilde{t}_2$, and $s_3 < \tilde{s}_3$, $t_3 < \tilde{t}_3$. We pick from α the transitions $(s_1, \text{compNext}_p, s'_1)$, $(t_1, \text{compNext}_q, t'_1)$, in such a way that there exists no other compNext_p , compNext_q between states s_1, \tilde{s}_1 respectively t_1, \tilde{t}_1 that sets NEXT_p respectively NEXT_q to i_1 . We can pick in a similar manner the transitions for jobs i_2, i_3 . From Lemmas 5.2, 5.3 and Definitions 5.1, 5.2 we have that $s_1 < s_2 < s_3$ and $t_1 < t_2 < t_3$. We will first prove that:

$$s_1.\text{DONE}_p \cup t_1.\text{DONE}_q \subseteq s_3.\text{DONE}_p \cap t_3.\text{DONE}_q$$

From algorithm KK_β we have that there exists in α transitions $(s_4, \text{setNext}_p, s'_4)$, $(t_4, \text{setNext}_q, t'_4)$ with $s'_4.\text{next}_p = i_2$, $t'_4.\text{next}_q = i_2$ and there exist no action $\pi_1 = \text{compNext}_p$, such that $s'_2 < \pi_1 < s'_4$, and no action $\pi_2 = \text{compNext}_q$, such that $t'_2 < \pi_2 < t'_4$. We will prove that $t_1 < s_4$ and $s_1 < t_4$. We start by proving that $t_1 < s_4$. In order to get a contradiction we assume that $s_4 < t_1$. From algorithm KK_β we have that there exists in α transition $(t_4, \text{gatherTry}_q, t'_4)$, with $t_4.Q_q = p$, and there exists no action $\pi_2 = \text{compNext}_q$, such that $t'_4 < \pi_2 < t_2$. We have that $s_4 < t_1 < t'_4 < t_2$ and $i_2 \notin t_2.\text{TRY}_q \cup t_2.\text{DONE}_q$. If $t_4.\text{next}_p = i_2$ we have a contradiction since $i_2 \in s_2.\text{TRY}_q$. If $t_4.\text{next}_q \neq i_2$ there exists an action $\pi_3 = \text{setNext}_p$ in α , such that $s_4 < \pi_3 < t_4$. If this $\pi_3 = \text{setNext}_p$ is preceded by transition $(s_5, \text{done}_p, s'_5)$ with $s_5.\text{NEXT}_p = i_2$, we have a contradiction since $i_2 \in t_4.\text{DONE}$ and $t_2.\text{DONE}_q$ is computed by gatherDone_q actions that are preceded by state t_4 , which results in $i_2 \in t_2.\text{DONE}_q$. If there exists no such transition we have again a contradiction state \tilde{s}_2 as defined by Definition 5.2 could not belong in α .

From the discussion above we have that $t_1 < s_4$. Thus $t_1.\text{DONE}_q < s_4.\text{DONE}$, moreover $s_3.\text{DONE}_p$ is computed by gatherDone_p actions that are preceded by state s_4 , from which we have that $t_1.\text{DONE}_q \subseteq s_3.\text{DONE}_p$. It is easy to see that $s_1.\text{DONE}_p \subseteq s_3.\text{DONE}_p$ holds, thus we have that $s_1.\text{DONE}_p \cup t_1.\text{DONE}_q \subseteq s_3.\text{DONE}_p$. With similar arguments as before, we can prove that $s_1.\text{DONE}_p \cup t_1.\text{DONE}_q \subseteq t_3.\text{DONE}_q$, which gives us that $s_1.\text{DONE}_p \cup t_1.\text{DONE}_q \subseteq s_3.\text{DONE}_p \cap t_3.\text{DONE}_q$.

Now it only remains to prove that:

$$|s_3.\text{DONE}_p \cup t_3.\text{DONE}_q| - |s_1.\text{DONE}_p \cup t_1.\text{DONE}_q| > m \cdot |q - p|$$

If $p < q$ from Lemma 5.1 we have that $|s_3.\text{DONE}_p \cap \overline{t_3.\text{DONE}_q}| > (q - p)m$ or $|\overline{s_3.\text{DONE}_p} \cap t_3.\text{DONE}_q| > (q - p)m$. Since $s_1.\text{DONE}_p \cup t_1.\text{DONE}_q \subseteq s_3.\text{DONE}_p \cap t_3.\text{DONE}_q$, we have that:

$$|s_3.\text{DONE}_p \cup t_3.\text{DONE}_q| - |s_1.\text{DONE}_p \cup t_1.\text{DONE}_q| > (q - p) \cdot m$$

If $q < p$ with similar arguments we have that:

$$|s_3.\text{DONE}_p \cup t_3.\text{DONE}_q| - |s_1.\text{DONE}_p \cup t_1.\text{DONE}_q| > (p - q) \cdot m$$

Combining the above we have:

$$|s_3.\text{DONE}_p \cup t_3.\text{DONE}_q| - |s_1.\text{DONE}_p \cup t_1.\text{DONE}_q| > m \cdot |q - p|$$

□

Lemma 5.5 *If $\beta \geq 3m^2$ there exists no execution $\alpha \in \text{execs}(\text{KK}_\beta)$ at which process p collided with process q in more than $2 \left\lceil \frac{n}{m|q-p|} \right\rceil$ states according to Definition 5.1.*

Proof. Let execution $\alpha \in \text{execs}(\text{KK}_\beta)$ be an execution at which process p collided with process q in at least than $2 \left\lceil \frac{n}{m|q-p|} \right\rceil + 1$ states. Let us examine the first $2 \left\lceil \frac{n}{m|q-p|} \right\rceil + 1$ such states. Let those states be $\tilde{s}_1 < \tilde{s}_2 < \dots < \tilde{s}_{2 \left\lceil \frac{n}{m|q-p|} \right\rceil + 1}$. From Lemma 5.2 we have that there exists states $s_1 < s_2 < \dots < s_{2 \left\lceil \frac{n}{m|q-p|} \right\rceil + 1}$ that enable the compNext_p actions and states $t_1 < t_2 < \dots < t_{2 \left\lceil \frac{n}{m|q-p|} \right\rceil + 1}$ that enable the compNext_q actions that lead to the collisions in states $\tilde{s}_1 < \tilde{s}_2 < \dots < \tilde{s}_{2 \left\lceil \frac{n}{m|q-p|} \right\rceil + 1}$. Then from Lemma 5.4 we have that $\forall i \in \left\{1, \dots, \left\lceil \frac{n}{m|q-p|} \right\rceil\right\}$:

$$|s_{2i+1}.\text{DONE}_p \cup t_{2i+1}.\text{DONE}_q| - |s_{2i-1}.\text{DONE}_p \cup t_{2i-1}.\text{DONE}_q| > m|q-p| \quad (7)$$

$$|s_{2i+1}.\text{DONE}_p \cup t_{2i+1}.\text{DONE}_q| - |s_1.\text{DONE}_p \cup t_1.\text{DONE}_q| > im|q-p| \quad (8)$$

$$|s_{2i+1}.\text{DONE}_p \cup t_{2i+1}.\text{DONE}_q| > im|q-p| \quad (9)$$

From 9 we have that:

$$\left| s_{2 \left\lceil \frac{n}{m|q-p|} \right\rceil + 1}.\text{DONE}_p \cup t_{2 \left\lceil \frac{n}{m|q-p|} \right\rceil + 1}.\text{DONE}_q \right| > m|q-p| \left\lceil \frac{n}{m|q-p|} \right\rceil \geq n \quad (10)$$

Equation 10 leads to a contradiction since $s_{2 \left\lceil \frac{n}{m|q-p|} \right\rceil + 1}.\text{DONE}_p \cup t_{2 \left\lceil \frac{n}{m|q-p|} \right\rceil + 1}.\text{DONE}_q \subseteq \mathcal{J}$ and $|\mathcal{J}| = n$. \square

Theorem 5.6 *If $\beta \geq 3m^2$ algorithm KK_β has work complexity $W_{\text{KK}_\beta} = O(nm \log n \log m)$.*

Proof. We start with the observation that in any execution α of algorithm KK_β , if there exists process p , job i , transition $(s_1, \text{done}_p, s'_1)$ and $j \in \{1, \dots, n\}$ such that $s_1.\text{POS}_p(p) = j$, $s_1.\text{NEXT}_p = i$, for any process $q \neq p$ there exists at most one transition $(t_1, \text{gatherDone}_q, t'_1)$ in α , with $t_1.Q_q = p$, $t_1.\text{POS}_q(p) = j$ and $t_1 \geq s_1$. Such transition performs exactly one read operation from the shared memory, one insertion at the set DONE_q and one removal from the set FREE_q , thus such a transition costs $O(\log n)$ work. Clearly there exist at most $m-1$ such transitions for each done_p . From Lemma 4.1 for all process there can be at most n actions done_p in any execution α of algorithm KK_β . Each done_p action performs one write operation in shared memory, one insertion at the set DONE_q and one removal from the set FREE_q , thus such an action has cost $O(\log n)$ work. Furthermore any done_p is preceded by $m-1$ gatherTry_p read actions that read the *next* array and each add at most one element to the set TRY_p with cost $O(\log n)$ and $m-1$ gatherDone_p read actions that do not add elements in the DONE_p set. Note that we have already counted the gatherDone_p read actions that result in adding jobs at the DONE_p set. Finally any done_p action is preceded by one compNext_p action. This action is dominated by the cost of $\text{rank}(\text{FREE}_p, \text{TRY}_p, i)$ function that has cost $O(m \log n)$, if the sets $\text{FREE}_p, \text{TRY}_p$ are represented with some efficient tree structure that allows insertion, deletion and search of an element in $O(\log n)$. We discussed at Section 3 what such tree structures could be. That gives us a total of bound of $O(nm \log n)$ work associated with the done_p actions.

If a process p collided with a process q in job i at state s , we have extra an extra compNext_p action, $m-1$ extra gatherTry_p read actions and insertions in the TRY_p set and $m-1$ gatherDone_p read actions that do not add elements in the DONE_p set. Thus each collision costs $O(m \log n)$ work. Since $\beta \geq 3m^2$ from Lemma 5.5 for two distinct processes p, q we have that in any execution α of algorithm KK_β there

exist less than $2 \left\lceil \frac{n}{m|q-p|} \right\rceil$ collisions. For process p if we count all such collisions with any other process q we get:

$$\sum_{q \in \mathcal{P} - \{p\}} 2 \left\lceil \frac{n}{m|q-p|} \right\rceil \leq 2(m-1) + \frac{2n}{m} \sum_{q \in \mathcal{P} - \{p\}} \frac{1}{|q-p|} \leq 2(m-1) + \frac{4n}{m} \sum_{i=1}^{\lceil \frac{m}{2} \rceil} \frac{1}{i} \leq 2(m-1) + \frac{4n}{m} \log m \quad (11)$$

If we count the total number of collisions for all the m processes we get that if $\beta \geq 3m^2$ in any execution of algorithm KK_β there can be at most $2m^2 + 4n \log m < 4(n+1) \log m$ collisions (since $n > \beta$). Thus collisions cost $O(nm \log n \log m)$ work. Finally any process p that fails may add in the work complexity less than $O(m \log n)$ work from its compNext_p action and from reads (if the process fails without performing a done_p action after its latest compNext_p action). So for the work complexity of algorithm KK_β if $\beta \geq 3m^2$ we have that $W_{\text{KK}_\beta} = O(nm \log n \log m)$. \square

6 An Asymptotically Work Optimal Algorithm

Here we demonstrate how to use algorithm KK_β with $\beta = 3m^2$ if $m = O(\sqrt[3]{n})$, in order to solve the at-most-once problem with effectiveness $n - O(m^2 \log n \log m)$ and work complexity $O(n + m^{(3+\epsilon)} \log n)$, for any constant $\epsilon > 0$, such that $1/\epsilon$ is a positive integer. We construct algorithm $\text{IterativeKK}(\epsilon)$ Fig. 2, that performs iterative calls to a variation of KK_β , which we call IterStepKK . $\text{IterativeKK}(\epsilon)$ has $3 + 1/\epsilon$ distinct *done* matrices in shared memory, with different granularities. One *done* matrix, stores the regular jobs performed, while the remaining $2 + 1/\epsilon$ matrices store *super-jobs*. Super-jobs are groups of consecutive jobs. From them, one stores super-jobs of size $m \log n \log m$, while the remaining $1 + 1/\epsilon$ matrices, store super-jobs of size $m^{1-i\epsilon} \log n \log^{1+i} m$ for $i \in \{1, \dots, 1/\epsilon\}$.

IterativeKK (ϵ) for process p :

```

00  sizep,1  $\leftarrow$  1
01  sizep,2  $\leftarrow$   $m \log n \log m$ 
02  FREEp  $\leftarrow$  map ( $\mathcal{J}$ , sizep,1, sizep,2)
03  FREEp  $\leftarrow$  IterStepKK (FREEp, sizep,2)
04  for( $i \leftarrow 1, i \leq 1/\epsilon, i++$ )
05    sizep,1  $\leftarrow$  sizep,2
06    sizep,2  $\leftarrow$   $m^{1-i\epsilon} \log n \log^{1+i} m$ 
07    FREEp  $\leftarrow$  map (FREEp, sizep,1, sizep,2)
08    FREEp  $\leftarrow$  IterStepKK (FREEp, sizep,2)
09  endfor
10  sizep,1  $\leftarrow$  sizep,2
11  sizep,2  $\leftarrow$  1
12  FREEp  $\leftarrow$  map (FREEp, sizep,1, sizep,2)
13  FREEp  $\leftarrow$  IterStepKK (FREEp, sizep,2)

```

Figure 2: Algorithm $\text{IterativeKK}(\epsilon)$: pseudocode

The algorithm IterStepKK is different from KK_β in three ways. First, all instances of IterStepKK work for $\beta = 3m^2$. Moreover IterStepKK has a termination flag in shared memory. This termination flag is initially 0 and is set to 1 by any process that decides to terminate. Any process that discovers that $|\text{FREE}_p \setminus \text{TRY}_p| < 3m^2$ in its compNext_p action, sets the termination flag to 1, computes new FREE_p and TRY_p set, returns the set $\text{FREE}_p \setminus \text{TRY}_p$ and terminates the current iteration. Any process p that checks if it is safe to perform a job, checks the termination flag first and if the flag is 1, the process instead of performing

the job, computes new FREE_p and TRY_p set, returns the set $\text{FREE}_p \setminus \text{TRY}_p$ and terminates the current iteration. Finally, IterStepKK takes as inputs the variable $size$ and a set SET_1 , such that $|\text{SET}_1| > 3m^2$, and returns the set SET_2 as output. SET_1 contains super-jobs of size $size$. In IterStepKK , with an action $\text{do}_{p,j}$ process p performs all the jobs of super-job j . IterStepKK performs as many super-jobs as it can and returns in SET_2 the super-jobs, which it can verify that no process will perform upon the termination of the algorithm IterStepKK . In $\text{IterativeKK}(\epsilon)$ we use also the function $\text{SET}_2 = \text{map}(\text{SET}_1, \text{size}_1, \text{size}_2)$, that takes the set of super-jobs SET_1 , with super-jobs of size $size_1$ and maps it to a set of super-jobs SET_2 with size $size_2$.

Theorem 6.1 *Algorithm $\text{IterativeKK}(\epsilon)$ has work complexity $W_{\text{IterativeKK}(\epsilon)} = O(n + m^{3+\epsilon} \log n)$ and effectiveness $E_{\text{IterativeKK}(\epsilon)}(n, m, f) = n - O(m^2 \log n \log m)$.*

Proof. In order to determine the effectiveness and work complexity of algorithm $\text{IterativeKK}(\epsilon)$, we compute the jobs performed by and the work spend in each invocation of IterStepKK . Moreover we compute the work that the invocations to the $\text{map}()$ function add. The first invocation to function $\text{map}()$ in line 02 can be completed by process p with work $O(\frac{n}{m \log n \log m} \log n)$, since process p needs to construct a tree with $\frac{n}{m \log n \log m}$ elements. This contributes for all processes $O(\frac{n}{\log m})$ work. From Theorem 5.6 we have that IterStepKK in line 03 has total work $O(n + \frac{n}{m \log n \log m} m \log n \log m) = O(n)$, where the first n comes from do actions and the second term from the work complexity of Theorem 5.6. Note that we count $O(1)$ work for each normal job executed by a do action on a super-job. That means that in the invocation of IterStepKK in line 03, do actions cost $m \log n \log m$ work. Moreover from Theorem 4.4 we have effectiveness $\frac{n}{m \log n \log m} - (3m^2 + m - 2)$ on the super-jobs of size $m \log n \log m$. From the super-jobs not completed, up to $m - 1$ may be contained in the TRY_p sets upon termination in line 03. Since those super-jobs are not added (and thus are ignored) in the output FREE_p set in line 03, up to $(m - 1)m \log n \log m$ jobs may not be performed by $\text{IterativeKK}(\epsilon)$. The set FREE_p returned by algorithm IterStepKK in line 03 has no more than $3m^2 + m - 2$ super-jobs of size $m \log n \log m$.

In each repetition of the loop in lines 04 – 09, the $\text{map}()$ function in line 07 constructs a FREE_p set with at most $O(m^{2+\epsilon}/\log m)$ elements, which costs $O(m^{2+\epsilon})$ per process p for a total of $O(m^{3+\epsilon})$ work for all processes. Moreover each invocation of IterStepKK in line 08 costs from Theorem 5.6 $O(3m^3 \log n \log m + m^{3+\epsilon} \log m) < O(m^{3+\epsilon} \log n)$ work, where the term $3m^3 \log n \log m$ is an upper bound on the work needed for the do actions on the super-jobs. From Theorem 4.4 we have that each output FREE_p set in line 08 has at most $3m^2 + m - 2$ super-jobs. Moreover from each invocation of IterStepKK in line 08 at most $m - 1$ super-jobs are lost in TRY sets. Those account for less than $(m - 1)m \log n \log m$ jobs in each iteration, since the size of the super-jobs in the iterations of the loop in lines 04 – 09 is strictly less than $m \log n \log m$.

When we leave the loop in lines 04 – 09, we have a FREE_p set with at most $3m^2 + m - 2$ super-jobs of size $\log n \log^{1+1/\epsilon} m$, which means that in line 12 function $\text{map}()$ will return a set FREE_p with less than $(3m^2 + m - 2)(\log n \log^{1+1/\epsilon} m)$ elements that correspond to jobs and not super-jobs. This costs for all processes a total of $O(m^3 \log m \log \log n \log \log m) < O(m^{3+\epsilon} \log n)$, since ϵ is a constant. Finally we have that IterStepKK in line 13 has from Theorem 5.6 work $O(m^3 \log^2 m \log \log n \log \log m) < O(m^{3+\epsilon} \log n)$, also from Theorem 4.4 it has effectiveness $(3m^2 + m - 2)(\log n \log^{1+1/\epsilon} m) - (3m^2 + m - 2)$.

If we add up all the work we have that $W_{\text{IterativeKK}(\epsilon)} = O(n + m^{3+\epsilon} \log n)$ since the loop in lines 04 – 09 repeats $1 + 1/\epsilon$ times and ϵ is a constant. Moreover for the effectiveness, we have that less than or equal to $(m - 1)m \log n \log m$ jobs will be lost in the TRY set at line 03. After that strictly less than $(m - 1)m \log n \log m$ jobs will be lost in the TRY sets of the iterations of the loop in lines 04 – 09 and less than $3m^2 + m - 2$ jobs will be lost from the effectiveness of the last invocation of IterStepKK in line 13. Thus we have that $E_{\text{IterativeKK}(\epsilon)}(n, m, f) = n - O(m^2 \log n \log m)$. \square

For any $m = O(\sqrt[3+\epsilon]{n/\log n})$, algorithm $\text{IterativeKK}(\epsilon)$ is work optimal and asymptotically effectiveness optimal.

6.1 An Asymptotically Optimal Work Complexity Algorithm for the Write-All Problem

```

WA_IterativeKK ( $\epsilon$ ) for process  $p$ :
00  $\text{size}_{p,1} \leftarrow 1$ 
01  $\text{size}_{p,2} \leftarrow m \log n \log m$ 
02  $\text{FREE}_p \leftarrow \text{map}(\mathcal{J}, \text{size}_{p,1}, \text{size}_{p,2})$ 
03  $\text{FREE}_p \leftarrow \text{WA\_IterStepKK}(\text{FREE}_p, \text{size}_{p,2})$ 
04 for ( $i \leftarrow 1, i \leq 1/\epsilon, i++$ )
05    $\text{size}_{p,1} \leftarrow \text{size}_{p,2}$ 
06    $\text{size}_{p,2} \leftarrow m^{1-i\epsilon} \log n \log^{1+i} m$ 
07    $\text{FREE}_p \leftarrow \text{map}(\text{FREE}_p, \text{size}_{p,1}, \text{size}_{p,2})$ 
08    $\text{FREE}_p \leftarrow \text{WA\_IterStepKK}(\text{FREE}_p, \text{size}_{p,2})$ 
09 endfor
10  $\text{size}_{p,1} \leftarrow \text{size}_{p,2}$ 
11  $\text{size}_{p,2} \leftarrow 1$ 
12  $\text{FREE}_p \leftarrow \text{map}(\text{FREE}_p, \text{size}_{p,1}, \text{size}_{p,2})$ 
13  $\text{FREE}_p \leftarrow \text{WA\_IterStepKK}(\text{FREE}_p, \text{size}_{p,2})$ 
14 for ( $i \in \text{FREE}_p$ )
15    $\text{do}_{p,i}$ 
16 endfor

```

Figure 3: Algorithm WA_IterativeKK (ϵ): pseudocode

Based on IterativeKK (ϵ) we construct algorithm WA_IterativeKK (ϵ) Fig. 3, that solves the *Write-All* problem [14] with work complexity $O(n + m^{(3+\epsilon)} \log n)$, for any constant $\epsilon > 0$, such that $1/\epsilon$ is a positive integer. From Kanellakis and Shvartsman [14] the Write-All problem for the shared memory model, consists of: “Using m processors write 1’s to all locations of an array of size n .” Algorithm WA_IterativeKK (ϵ) is different from IterativeKK (ϵ) in two ways. It uses a modified version of IterStepKK, that instead of returning the $\text{FREE}_p \setminus \text{TRY}_p$ set upon termination returns the set FREE_p instead. Let us name this modified version WA_IterStepKK. Moreover in WA_IterativeKK (ϵ) after line 13 process p , instead of terminating, executes all tasks in the set FREE_p . Note that since we are interested in the Write-All problem, when process p performs a job i with action $\text{do}_{p,i}$, process p just writes 1, in the i -th position of the Write All array $wa[1, \dots, n]$ in shared memory.

Theorem 6.2 *Algorithm WA_IterativeKK (ϵ) solves the Write-All problem with work complexity $W_{\text{WA_IterativeKK}(\epsilon)} = O(n + m^{3+\epsilon} \log n)$.*

Proof. (of Theorem 6.2) We prove this with similar arguments as in the proof of Theorem 6.1. As in the proof of Theorem 6.1 after each invocation of WA_IterStepKK the output set FREE_p has less than $3m^2 + m - 1$ super-jobs, from Theorem 4.4. The difference is that now we don’t leave jobs in the TRY_p sets, since we are not interested in maintaining the at-most-once property between successive invocations of the WA_IterStepKK algorithm. Since after each invocation of WA_IterStepKK the output set FREE_p has the same upper bound on super-jobs as in IterativeKK (ϵ), with similar arguments as in the proof of Theorem 6.1, we have that at line 13 the total work performed by all processes is $O(n + m^{3+\epsilon} \log n)$. Moreover from Theorem 4.4 the output FREE_p set in line p has less $3m^2 + m - 2$ jobs. This gives us for all processes a total work of $O(m^{3+\epsilon})$ for lines the loop in lines 14 – 16. After the loop in lines 14 – 16 all jobs have been performed, since we left no TRY sets behind, thus algorithm WA_IterativeKK (ϵ) solves the Write-All problem with work complexity $W_{\text{WA_IterativeKK}(\epsilon)} = O(n + m^{3+\epsilon} \log n)$. \square

For any $m = O(\sqrt[3+\epsilon]{n/\log n})$, algorithm WA_IterativeKK (ϵ) is work optimal.

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